Relative extrema and second order conditions

Compute the relative extrema of the following function and verify the second order conditions.

$$z = 2x^3 - 9x^2 + 12x + 2y^3 - 3y^2 + 1$$

Solution

We compute the first order conditions:

$$z'_x = 6x^2 - 18x + 12 = 0$$

$$z'_y = 6y^2 - 6y = 0$$

From the first equation, we get 2 roots $x = 1$ and $x = 2$. From the second equation, we also get two roots: $y = 0$ and $y = 1$. With this, there are 4 possible combinations since the two equations are independent of each other: $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$. We compute the second derivatives for the second order conditions:

$$z''_{xx} = 12x - 18$$

$$z''_{yy} = 12y - 6$$

$$z''_{yx} = z''_{xy} = 0$$

We compute the determinant of the Hessian:

$$|H| = \begin{vmatrix} 12x - 18 & 0 \\ 0 & 12y - 6 \end{vmatrix} = (12x - 18)(12 - 6y)$$

Now, we insert the values of the various critical points.

- With the point $(1, 0)$. $|H| = 36 > 0$. Since $f''_{xx} = -6 < 0$ we have a relative maximum.
- With the point $(1, 1)$. $|H| = -36 < 0$. Saddle point.
- With the point $(2, 0)$. $|H| = -36 > 0$. Saddle point.
- With the point $(2, 1)$. $|H| = 36 > 0$. Since $f''_{xx} = 6 > 0$ we have a relative minimum.